Chapter 8

Representations of two compact groups.

8.1 Representations of the permutation group

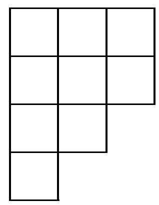
Permutations on n symbols is the set of one to one mappings σ of a set $X = \{x_1, x_2, \ldots, x_n\}$ of n elements on to itself. It is a finite group G with $n!$ elements. Given $\sigma \in G$ we can look at the orbits of $\sigma^n x$ and it will partition the speace X into orbits $A_1, A_2, \ldots, A_k$ consisting of $n_1 \geq \ldots \geq n_k$ points so that $n = n_1 + \cdots + n_k$. If $\hat{\sigma} = s\sigma s^{-1}$ is conjugate to σ then the orbits of $\hat{\sigma}$ will be $sA_1, sA_2, \ldots, sA_k$ so that the partition $n_1 \geq \ldots \geq n_k$ of n into k numbers will be the same for σ and $\hat{\sigma}$. Conversely if σ and $\hat{\sigma}$ have the same partition of n, then the orbits $A_1, \ldots, A_k$ and $B_1, \ldots, B_k$ can be so arranged that the corresponding orbits have the same cardinality. We can find $s \in G$ that maps $A_i \to B_i$ in a one-to-one and onto manner and we can reduce the problem to the case where $A_i = B_i$ for every i. We can now relablel the points with in each A_i i.e. find a permutation of n_i elements, such that both σ and $\hat{\sigma}$ look the same on each A_i. We can now prove the following theorem.

Theorem 8.1. *The number of distinct inequivalent irreducible representations of G is the same as the number of distinct partitions $\mathcal{P}(n)$ of the integer n.*

Proof. The space $\mathcal{U}$ of functions u that are invariant under conjugation i.e satisfy $u(h^{-1}gh) = u(g)$ is spanned by characters and its dimension equals the number of conjugacy classes, i.e the number of distinct sequences $n_1 \geq$

$n_2 \geq \cdots \geq n_k > 0$ such that $n = n_1 + \cdots + n_k$. This is the same as the number of distinct partitions of n. $\square$

We will now construct a distinct irreducible representation for each partition of n. Given a partition λ of n into $n_1 \geq n_2 \geq \cdots \geq n_k$, we associate a diagram called the Young diagram corresponding to λ. It looks like



when $n = 9$ and the partition is $3, 3, 2, 1$. Each set in the partition is a row consisting of squares whose number is the number of elements in that set. The rows are arrnged in decreasing order of their size and aligned on the left. A Young tableau $\mathbf{t}$ is a diagram λ with the boxes filled in arbitrarily by the numbers $1, 2, \cdots, 9$, like

4	1	8
5	2	7
3	6	
9		

The rows of the array are of length $n_1 \geq \cdots \geq n_k$. For any diagram there are $n!$ tableaux. The tableaux $\mathbf{t}$ above has rows $[4, 1, 8]$, $[5, 2, 7]$ $[3, 6]$ and $[9]$. A tabloid is when the order of entries with in a row is not relevant. The tabloid $\tau = \{\mathbf{t}\}$ consists of $\{1, 4, 8\}, \{2, 5, 7\}, \{3, 6\}, \{9\}$. There are $\frac{n!}{n_1! \cdots n_k!}$ tabloids corresponding to a diagram λcorresponding to the partition $n_1 \geq n_2 \geq \cdots \geq n_k$. The subgroup $C_{\mathbf{t}}$ of the permutation group consists of permutations within each column. In our case it consists of $4! \times 3! \times 2! = 288$ elements. An arbitrary permutation of $4, 5, 3, 9$, an arbitrary permutation of $1, 2, 6$ and one of $8, 7$. For any permutation s, $\sigma(s) = \pm 1$ is the sign of the permutation. We define an abstract inner product space $V - \lambda$ of dimension $\frac{n!}{n_1! \cdots n_k!}$ with the orthonormal basis e_τ as τ varies over all the tabloids of the diagram λ. We define $n!$ vectors $f_{\mathbf{t}}$ in V_λ as $\mathbf{t}$ varies over the set $\mathbf{T}$ of tableau of the diagram λ. $\{\mathbf{t}\}$ is the tabloid that is the equivalence class of all the tableaux that are obtained by permutations with in rows of the tableaux $\mathbf{t}$. The group element, namely a permutation s acts naturally on the set of tableaux $\mathbf{T}$. If two tableaux $\mathbf{t}_1$ and $\mathbf{t}_2$ belong to the same tabloid so do $s\mathbf{t}_1$ and $s\mathbf{t}_2$. Therefore s acts on the set $\mathcal{T}$ of tabloids as well. Define

$$f_{\mathbf{t}} = \sum_{s \in C_{\mathbf{t}}} \sigma(s) e_{s\{\mathbf{t}\}}$$

The $\{f_{\mathbf{t}} : \mathbf{t} \in \mathbf{T}\}$ may not be linearly independent and the span of $\{f_{\mathbf{t}}\}$ is denoted by W. One defines a representation of $\pi_\lambda(g)$ of the permutation group on V_λ and W_λ corresponding to the diagram λ by defining

$$\pi_\lambda(g) f_{\mathbf{t}} = g f_{\mathbf{t}} = \sum_{s \in C_{\mathbf{t}}} \sigma(s) e_{gs\{\mathbf{t}\}}$$

Theorem 8.2. *Each π_λ is irreducible. For two distinct diagrams they are inequivalent. We therefore have all the representations.*

Proof is broken up into lemmas.

Lemma 8.3.

$$\pi_\lambda(g) f_{\mathbf{t}} = f_{g\mathbf{t}}$$

Proof.

$$\begin{aligned}
\sum_{s \in C_{\mathbf{t}}} \sigma(s) e_{gs\{\mathbf{t}\}} &= \sum_{s \in C_{\mathbf{t}}} \sigma(s) e_{gsg^{-1}g\{\mathbf{t}\}} \\
&= \sum_{gsg^{-1} \in C_{g\mathbf{t}}} \sigma(s) e_{gsg^{-1}g\{\mathbf{t}\}} \\
&= \sum_{s' \in C_{g\mathbf{t}}} \sigma(s') e_{s'g\mathbf{t}} = f_{g\mathbf{t}}
\end{aligned}$$

□

Lemma 8.4. *Suppose λ, μ are two different diagrams, $\mathbf{t}$ a λ-tableaux and $\mathbf{t}^*$ a μ-tableaux. Suppose*

$$\sum_{s \in C_{\mathbf{t}}} \sigma(s) e_{s\{\mathbf{t}^*\}} \neq 0$$

Then $n_1 \geq m_1, n_1 + n_2 \geq m_1 + m_2, \cdots$ where $n_1, \ldots, n_k$ and $m_1, \ldots, m_\ell$ are the two partitions corresponding to λ and μ. We say then that $\lambda \geq_1 \mu$. If $\lambda = \mu$ then the sum is $\pm f_{\mathbf{t}}$.

Proof. Suppose that two elements a, b are in the same row of $\mathbf{t}^*$ and in the same column of $\mathbf{t}$. Then the permutation $p = \{a \leftrightarrow b\}$ is in $C_{\mathbf{t}}$, $e_{sp\{\mathbf{t}\}} = e_{s\{\mathbf{t}\}}$, with $\sigma(sp) + \sigma(s) = 0$. So the sum adds up to 0 which is ruled out. Hence, no two elements in the same row of $\mathbf{t}^*$ can be in the same column of $\mathbf{t}$. In particular $\mathbf{t}$ must have at least as many columns as the number of elements in the first row of $\mathbf{t}^*$ proving $n_1 \geq m_1$. A variant of this argument proves $\lambda \geq_1 \mu$. Suppose now that $\lambda = \mu$. Again all the elements in any row of $\mathbf{t}^*$ appear in different columns of $\mathbf{t}$. Then there is a permutation $s^* \in C_{\mathbf{t}}$ such that $s^*\mathbf{t} = \mathbf{t}^*$. The sum is unaltered if we replace $\mathbf{t}$ by $s^*\mathbf{t}$ except for the factor of $\sigma(s^*) = \pm 1$. □

Lemma 8.5. *Let $u \in V$ corresponding to a diagram λ. Let $\mathbf{t}$ be any λ-tableau. Then*

$$\sum_{s \in C_{\mathbf{t}}} \sigma(s)\pi_\lambda(s)u = cf_{\mathbf{t}}$$

Proof. u is a linear combination of $e_{\{\mathbf{t}^*\}}$ for different tabloids $\{\mathbf{t}^*\}$ corresponding to λ . Each one from the previous lemma yields $c(\mathbf{t}^*)f_{\mathbf{t}}$ with $c(\mathbf{t}^*) = 0, \pm 1$. Add them up! □

Let us define

$$A_{\mathbf{t}} = \sum_{s \in C_t} \sigma(s)\pi_\lambda(s)$$

acting on V. We already have an inner product that makes $\pi(s)$ orthogonal.

$$\begin{aligned} < A_{\mathbf{t}}u, v > &= \sum_{s \in C_{\mathbf{t}}} \sigma(s) < \pi(s)u, v > \\ &= \sum_{s \in C_{\mathbf{t}}} \sigma(s) < u, \pi(s^{-1})v > \\ &= \sum_{s \in C_{\mathbf{t}}} \sigma(s) < u, \pi(s)v > \\ &=< u, A_{\mathbf{t}}v > \end{aligned}$$

because $\sigma(s^{-1}) = \sigma(s)$.

Lemma 8.6. *If U_λ is any invariant subspace of V then either $U_\lambda \supset W_\lambda$ or $U_\lambda \perp W_\lambda$. This proves the irreducibility of W_λ.*

Proof. Suppose for some $u \in U_\lambda \subset V_\lambda$ and $\mathbf{t}$ is a λ-tableau. We saw that $A_{\mathbf{t}} u = c_{\mathbf{t}} f_{\mathbf{t}}$ for some constant $c_{\mathbf{t}}$. Suppose for some $u \in U_\lambda$ and $\mathbf{t}$, $c_{\mathbf{t}} \neq 0$. Then $f_{\mathbf{t}} = \frac{1}{c_{\mathbf{t}}} A_{\mathbf{t}} u \in U_\lambda$. Since $\pi(g) f_{\mathbf{t}} = f_{g\mathbf{t}}$ and $\{f_{g\mathbf{t}}\}$ span W_λ, it follows that $W_\lambda \subset U_\lambda$. If $c_{\mathbf{t}} = 0$ for all u and $\mathbf{t}$, $A_{\mathbf{t}} u = 0$ and it follows that

$$< u, f_{\mathbf{t}} >= \pm < u, A_{\mathbf{t}} f_{\mathbf{t}} >= \pm < A_{\mathbf{t}} u, f_{\mathbf{t}} >= 0$$

and $u \in W^\perp$. □

Lemma 8.7. *Let T intertwine the representations on V_λ and V_μ. Suppose W_λ is not contained in* Ker T. *Then $\lambda \geq_1 \mu$.*

Proof. Ker T is invariant under $\pi(g)$ and if it does not contain $W\lambda$ it is orthogonal to it.

$$0 \neq T f_{\mathbf{t}} = T A_{\mathbf{t}} f_{\mathbf{t}} = A_{\mathbf{t}} T f_{\mathbf{t}}$$

$T f_{\mathbf{t}}$ is a combination of $e_{\{\mathbf{t}\}}$ of μ-tableau $\{\mathbf{t}\}$. So at least one of them $A_{\mathbf{t}} e_{\{\mathbf{t}\}}$ is nonzero forcing $\lambda \geq_1 \mu$. □

Lemma 8.8. *If $T \neq 0$ intertwines W_λ and W_μ, then $\lambda = \mu$.*

Proof. Extend T by making it 0 on $(W_\lambda)^\perp$ and we see that $\lambda \geq_1 \mu$. The argument is symmetric. □

8.2 Representations SO(3)

We will consider the irreducible representations of the group G of rotations in R^3. These are orthogonal transformations of determinant 1, i.e. that preserve orientation. An element $g \in G$ is represented as the matrix

$$\begin{bmatrix} t_{1,1}(g) & t_{1,2}(g) & t_{1,3}(g) \\ t_{2,1(g)} & t_{2,2}(g) & t_{2,3}(g) \\ t_{3,1(g)} & t_{3,2}(g) & t_{3,3}(g) \end{bmatrix}$$

There is the trivial representation $\pi_0(g) \equiv I$. Then there is a natural three dimensional representation where $\pi_1(g) = t(g) = \{t_{i,j}(g)\}$ and it can be viewed as a unitary representation in $\mathcal{C}^3$. This representation is irreducible and faithful, i.e. it separates points of G.

As we saw in the general theory, the characters can be used to identify the irreducible representations. It helps to know what the conjugacy classes are. Given two orthogonal matrices g_1 and g_2, when can we find a g such that $gg_1g^{-1} = g_2$? The eigen values of g_1 are $1, e^{\pm i\theta_1}$ and therefore in order for g_1 and g_2 to be mutually conjugate we neeed $\theta_1 = \pm\theta_2$ or $cos\theta_1 = \cos\theta_2$. Conversely one can show that that if g_1 and g_2 have the same eigenvalues then they are indeed conjugate. If we use a g to align the eigenspace correponding to 1, then we need to show essentially that rotation by θ and $-\theta$ are conjugate. We can use the matrix

$$\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

to achieve this.

We will use the infinitesimal method to study irreducible representations. If $A = \{a_{i,j}\}$ is a real skewsymmetric matrix then $g_t = e^{tA}$ defines a one parameter curve in G, and if π is a unitary representation on a complex vector space V, then $U_t = \pi(g_t) = e^{it\sigma(A)}$ for some skew symmetric $\sigma(A)$. This way we get a map $A \to \sigma(A)$ from the space of real skewsymmetric 3×3 matrices into complex skewhermitian matrices on V.

The way to understand this map is to think of G as three dimensional manifold and the vector space of real skewsymmetric 3×3 matrices as the tangent space at e. In fact there are global vector fields acting on functions defined on G corresponding to any skew symmetric A,

$$(X_A)f(g) = \frac{d}{dt} f(ge^{tA})|_{t=0}$$

Then

$$\sigma(A) = (X_A)\pi(e)$$

and from the representation property

$$(X_A)\pi(g) = \pi(g)\sigma(A)$$

$$X_A X_B = \sigma(A)\sigma(B)$$

The Poisson bracket $[X_A, X_B] = X_A X_B - X_B X_A$ is to equal $X_{[AB-BA]}$ and we get this a way a representation σ of the "Lie Algebra" of 3×3 skewsymmetric matices in the space of skewhermitian trnasfromations on V. Moreover $\sigma([A, B]) = [\sigma(A), \sigma(B)]$. G acts irreducibly on V if and only if $\sigma(A)$ acts irreducibly. We pick a basis A_1, A_2, A_3 where

$$A_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \quad A_2 = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \quad A_3 = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Let us note that

$$[A_1, A_2] = -A_3, [A_2, A_3] = -A_1, [A_3, A_1] = -A_2$$

If we define $\sigma(A_1) = H$ and $Z_1 = \sigma(A_2) + i\sigma(A_3), Z_2 = \sigma(A_2) - i\sigma(A_3)$, we can calculate

$$[H, Z_1] = \sigma([A_1, A_2]) + i\sigma([A_1, A_3]) = -\sigma(A_3) + i\sigma(A_2) = iZ_1$$
$$[H, Z_2] = \sigma([A_1, A_2]) - i\sigma([A_1, A_3]) = -\sigma(A_3) - i\sigma(A_2) = -iZ_2$$

H being skewhermitian on V, it has purely imaginary eigenvalues and a complete set of eigenvectors. Let $V = \oplus_\lambda V_{i\lambda}$ be the decomposition of V into eigenspaces of H. Moreover $e^{2\pi H} = \pi(e^{2\pi A_1}) = \pi(e) = I$ The values λ are therefore all integers. If $Hv = i\lambda v$, then $HZ_1 v = Z_1 H v + [H, Z_1]v = i\lambda Z_1 v + iZ_1 v = i(\lambda + 1)Z_1 v$. Therefore Z_1 maps $V_{i\lambda} \to V_{i(\lambda+1)}$ and similarly Z_2 maps $V_{i\lambda} \to V_{i(\lambda-1)}$. It is clear that if we start with some $v_0 \in V_{i\lambda}$ then $v_0, \{Z_1^k v_0 : k \geq 1\}, \{Z_2^k v_0 : k \geq 1\}$ are all mutually orthogonal. Since the space is finite dimensional, $Z_1^r v_0 = Z_2^s v_0 = 0$ for some r, s. If we take r, s to be the smallest such values, then the subspace generated by them has dimension $r + s - 1$ and is invariant under H, Z_1, Z_2. Since the representation is irreducible, this must be all of V. Another piece of information is that H and $-H$ are conjugate. The set of λ's is therefore symmetric around the origin. Hence V is odd dimensional and is $\{\lambda\} = \{-k, \ldots, 0, \ldots, k\}$ for some integer $k \geq 0$. This exhausts all possible irreducible representations in the infinitesimal sense and therefore the set of irreducible representations of G cannot be larger. The character of such a representation if it exists is seen to be

$$\chi_k(g) = \hat{\chi}_k(\theta) = \sum_{j=-k}^{k} \exp[i\,j\theta]$$

where $1, e^{\pm i\theta}$ are the eigenvalues of g. We will try construct them as the natural action of G on the space of homogeneous harmonic polynomials of degree k. This dimension is calculated as $\frac{(k+1)(k+2)}{2} - \frac{k(k-1)}{2} = 2k+1$. H which is the infinitesimal rotation around x-axis is calculated as

$$H = z\frac{\partial}{\partial y} - y\frac{\partial}{\partial z}$$

The polynomials $p_k^{\pm} = (y \pm iz)^k$ are harmonic in two and therefore three variables and $Hp_k^{\pm} = \pm ikp_k^{\pm}$. Therefore this representation has the eigenvlaues $\pm ik$ for H and cannot be decomposed totally in terms of representations of dimension $(2k-1)$ or less. On the other hand its dimension is only $(2k+1)$. This is it.

Since we know that $\int_G \chi_k(g)\chi_\ell(g)dg = \delta_{k,\ell}$ it is convenient to determine the weight $w(\theta)$ on $[0,\pi]$ such that it is the probability density of $\theta(g)$ of a random g uniformly distributed over G. Then

$$\int_0^{\pi} \hat{\chi}_k(\theta)\hat{\chi}_\ell(\theta)w(\theta)d\theta = \delta_{k,\ell}$$

In particular, since $\hat{\chi}_1(\theta) \equiv 1$, for $k \geq 2$

$$\int_0^{\pi} [\hat{\chi}_k(\theta) - \hat{\chi}_{k-1}(\theta)]w(\theta)d\theta = 0$$

or

$$w(\theta) = a + b\cos\theta$$

Normalization of $\int_0^{\pi} w(\theta)d\theta = 1$ gives $a = \frac{1}{\pi}$. The orthogonality relation $\int_0^{\pi} 1.(1 + 2\cos\theta)w(\theta)d\theta = 0$ provides $a + b = 0$ or

$$w(\theta) = \frac{1 - \cos\theta}{\pi}$$